1. (18 points) An Easy Practice

A body of mass \( m = 4 \text{ kg} \) attached to the end of a massless rod of length \( l = 0.5 \text{ m} \) is rotating counterclockwise in the vertical plane with constant angular velocity \( \omega = 1 \text{ s}^{-1} \) as shown in the picture below.

The answer to each question has three parts, value (1 point), unit (1 point), and direction (1 point). If for a particular question you believe the direction to be undefined or irrelevant, please leave it blank.

(a) What is the body’s momentum in the topmost point of the trajectory?

(b) What is the body’s acceleration in the leftmost point of the trajectory?

(c) What is the kinetic energy of the body in the bottommost point of the trajectory?

(d) What is the difference between gravitational potential energies in the topmost and bottommost points?

(e) What is the tension force the rod exerts on the body in the bottommost point of the trajectory?

(f) What is the work done by the tension force over one revolution?

1. Solution

This problem should be straightforward. The motion is a uniform circular motion – so angular velocity is constant, and the magnitude of the linear velocity is constant. Therefore,

(a) \( p = mv = m\omega l = 2 \text{ kg} \cdot \text{m/s}, \) pointing left

(b) Acceleration is centripetal: \( a_{\text{centripetal}} = \omega^2 l = 0.5 \text{ m/s}^2, \) pointing downward

(c) Since the magnitude of the velocity is constant, the kinetic energy does not depend on the point in the trajectory; \( T = \frac{mv^2}{2} = \frac{m\omega^2 l^2}{2} = 0.5 \text{ J} \)

(e) \( \Delta U = 2mgl = 49 \text{ J} \)

(f) The force does depend on position. At each point acceleration is centripetal and equal in magnitude to (b). In the bottommost point, equation of motion in \( Y \) direction is

\[
ma = T - mg
\]

(acceleration \( a \) points towards the rotation axis, \( i.e. \) upwards). Therefore

\[
T = m(g + a) = m(g + \omega^2 l) = 40 \text{ N}
\]

(h) The work is clearly zero since both the kinetic energy and the potential energy of the body after one revolution are the same. Note however that the tension force is not always normal to the direction of the motion: clearly at the left- and rightmost points on the trajectory the tension force has to have vertical component to compensate the gravitational force. The work done by the tension force is positive on the right side of the trajectory and negative on the left side, for the total integral of zero.
2. (20 points) \( g \) is for Gymnastics

My kids’ gym has a trampoline that I helped set up. The trampoline is basically a sheet of non-stretchable fabric, connected to a rectangular frame by stiff springs. It behaves more or less like a vertical spring (though a person on top of the trampoline is obviously not connected to it), and to first approximation obeys the Hooke’s law. I noticed that when I stand in the middle of the trampoline, it sags by about 6 inches (15 cm). My mass is \( m_y = 65 \) kg. The trampoline is set up over a pit (see picture), and the maximum safe distance the trampoline can be stretched downward is about 3 ft (90 cm).

(a) (15 points) How high (relative to ground) can a child of mass \( m_c = 50 \) kg jump after he stretches the trampoline maximally and bounces straight up?

(b) (5 points) What is the maximum acceleration he experiences?

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2. Solution

(a) This problem is best solved using the concept of conservation of energy, though you can also work it out using the work-energy theorem. There are two types of potential energy involved: the elastic potential energy of the stretched spring (trampoline), and the gravitational potential energy. Both can be converted into kinetic energy.

We will compare two points: the initial position would be the child on a maximally stretched trampoline, and the final position would be at the top of the flight. At the initial position, the total mechanical energy contains potential energy of the spring (trampoline) and potential energy due to gravity:

\[
E_i = \frac{kd_{\text{max}}^2}{2} - m_c g d_{\text{max}}
\]

where \( d_{\text{max}} = 90 \) cm is the maximum stretch of the trampoline. Notice that there is no kinetic energy term (because when the trampoline stretches maximally, the child is momentarily at rest, before being pushed up).

Also note the minus sign in front of the gravitational potential energy: this is because the child is below the ground level, which we use as origin of the coordinate system \((y = 0)\).

At the top of his trajectory, the child’s entire energy is contained in the gravitational piece:

\[
E_f = m_c g h_{\text{max}}
\]

We can now find the maximum elevation from conservation of energy (we ignore air resistance and any other friction forces):

\[
m_c g h_{\text{max}} = \frac{kd_{\text{max}}^2}{2} - m_c g d_{\text{max}}

h_{\text{max}} = \frac{kd_{\text{max}}^2}{2m_c g} - d_{\text{max}} \quad (1)
\]

But we do not know the spring constant \( k \). We can find it from the initial conditions: the balance of forces when the adult (me) stands in the center of the trampoline. In this case, the vertical acceleration is zero, which means the weight of the adult \( m_y g \) is balanced by the spring force of the trampoline \( kd \), where \( d = 15 \) cm:

\[
m_y g = kd \Rightarrow k = \frac{m_y g}{d} = 4.3 \text{ kN/m}
\]

Plugging this into Eq. (1), we finally find the elevation of the child:

\[
h_{\text{max}} = \frac{d_{\text{max}}^2}{2d} m_y - d_{\text{max}} = 2.6 \text{ m}
\]

That’s a pretty decent elevation.

(b) The child experiences maximum acceleration when the trampoline is stretched maximally. At that point there are two forces acting on him: the upward force from the trampoline, and the downward gravity force.

\[
a = \frac{kd_{\text{max}} - m_c g}{m_c} = g \left( \frac{m_y d_{\text{max}}}{m_c d} - 1 \right)
\]

\[
= 7 g = 70 \text{ m/s}^2
\]

That’s some serious G-force.

3. (25 points) Keep your head on a swivel

It is Game 7 of the 2000 NHL Eastern Conference Finals between Philadelphia Flyers and New Jersey Devils. Eric Lindros, a 244 lb (110 kg) forward for the
Flyers, is skating with a puck into the New Jersey zone, keeping his head down (big mistake). His speed is \( v_L = 10 \text{ m/s} \). At the blue line, he is met by the Devils’ defenseman Scott Stevens, who weighs 220 lbs (100 kg). Just before the collision, Stevens accelerates to \( v_S = 11 \text{ m/s} \), and meets Lindros head-on, planting his shoulder squarely into poor Eric’s chest.

(a) (10 points) Assuming the collision is elastic (i.e. no bones were crushed), what is Eric Lindros’ velocity after the collision?

(b) (5 points) If the collision lasted \( \Delta t = 0.2 \text{ sec} \), estimate the average force experienced by Lindros, and average acceleration of his body during the collision.

(c) (10 points) Approximating Lindros’ body as a stiff (he lost conscientious after the collision) solid rod of height \( h = 6'4'' = 1.93 \text{ m} \) and assuming that his skates never left the ice after the collision, estimate the velocity with which his head hit the ice.

3. Solution

This problem is based on a real incident, which was one of the deciding factors in tilting that Game 7 in New Jersey favor (they won, and went on to take the Stanley Cup from the Dallas Stars in the final).

(a) Momentum is a vector quantity. Let’s point \( X \) axis in the direction of Lindros’ initial velocity. With this choice, the \( X \) projection of Lindros’ velocity is \( v_{Lx} = v_L = 10 \text{ m/s} \), and the \( X \) projection of Stevens’ velocity is \( v_{Sx} = -v_S = -11 \text{ m/s} \). The Lindros’ momentum projection is \( p_{Lx} = m_L v_L \), and Stevens’ momentum projection is \( p_{Sx} = -m_S v_S \). The total momentum along \( X \) direction is

\[
P_x^{\text{tot}} = p_{Lx} + p_{Sx} = m_L v_L - m_S v_S = 0 \quad (2)
\]

The numbers work out just right: this is the center-of-mass frame! So the calculations of the scattering process should be fairly simple mathematically.

The kinetic energy of the players is

\[
K^{\text{tot}} = \frac{m_L v_L^2}{2} + \frac{m_S v_S^2}{2} = 11.6 \text{ kJ} \quad (3)
\]

The collision is head-on, so the velocity of each player has changed direction. Let’s label Lindros’ velocity projection after collision \( v'_{Lx} = -v_L \), and the velocity of Stevens after the collision is \( v'_{Sx} = +v_S \) (\( v'_L \) and \( v'_S \) are speeds, or magnitudes of the player velocities). Since the collision is elastic, both total momentum and total kinetic energy are conserved:

\[
P_x^{\text{tot}} = 0 = -m_L v'_L + m_S v'_S \quad (4)
\]

\[
K^{\text{tot}} = \frac{m_L v'_L^2}{2} + \frac{m_S v'_S^2}{2} \quad (5)
\]

To solve this system of equations, we need to express \( v'_S \) in terms of \( v'_L \) using Eq. (4):

\[
v'_S = v'_L \frac{m_L}{m_S}
\]

and then plug it into Eq. (5):

\[
K^{\text{tot}} = \frac{m_L v'_L^2}{2} \left(1 + \frac{m_L}{m_S}\right) \quad (6)
\]

We can now plug in \( K^{\text{tot}} \) from Eq. (3) and solve Eq. (6) for \( v'_L \). On the other hand, we can realize that Eq. (6) is very similar to the expression for kinetic energy in Eq. (3). That is, if we solved for Lindros’ speed after collision \( v'_L \) in terms of his speed before collision \( v_L \), we’d find that they are equal:

\[
v'_L = v_L
\]

This means that his velocity is

\[
v'_{Lx} = -v_L = -10 \text{ m/s}.
\]

This is a very general result for the center of mass system (which we derived in class): in any two-body elastic collision, the center-of-mass speed of each object before and after the collision is preserved!

(b) We can now find the force that Eric experienced during the collision from momentum-impulse relation:

\[
\Delta p_{Lx} = m_L v'_{Lx} - m_L v_{Lx} = -2m_L v_L = F_x \Delta t
\]

The force is

\[
F_x = \frac{-2m_L v_L}{\Delta t} = -11 \text{kN} (!)
\]

During the collision, Eric experienced acceleration of

\[
a_x = \frac{F_x}{m_L} = -100 \text{ m/s}^2
\]
or 10g! That’s the acceleration top fighter pilots experience during acrobatic maneuvers. Not surprisingly, Lindros, Flyers’ best but concussion-prone forward, was knocked out of the game. By some accounts, he was unconsolent before he fell down to the ice... But the hit was clean and legal. Hockey is the fastest team sport, and so collisions can be violent! That’s why you can often hear hockey announcers use the phrase “Keep your head on a swivel!” — that is, look out, especially when the likes of Stevens are lurking around the blue line!

(c) After the collision, Lindros’ center of mass is moving with velocity $v'_{Lx}$. At the same time, his body is rotating under the influence of the gravitational torque. Since we know the skates never leave the ice, it is easiest to consider the rotation around the bottom point (i.e., the skates). When he hits the ice, the body is horizontal. That means the rotational velocity is vertical. Suppose his body at that point has an angular velocity $\omega$. Then the rotational velocity of the head is $v_{yh} = \omega h$ (directed straight down) and the total velocity is a quadrature sum of the vertical velocity $v_{yh}$ and the horizontal velocity $v'_{Lx}$ (which does not change, since there is no friction):

$$v_h = \sqrt{v^2_{Lx} + v^2_{yh}}$$

So we just need to find $\omega$. We will do that using the conservation of energy. When Lindros hits the ice, the total energy of his body is a sum of translational and rotational kinetic energies:

$$K_{\text{final}} = \frac{mv'^2_{Lx}}{2} + \frac{I\omega^2}{2}$$

On the other hand, right after the collision, the body had the translational kinetic energy and also the potential energy:

$$K_{\text{initial}} = \frac{mv'^2_{Lx}}{2} + mg\frac{h}{2}$$

The gravitational potential energy is computed at the location of the center of mass. The translational kinetic energy terms in Eq. (7) and Eq. (8) are obviously the same, so we can find the angular velocity:

$$\omega = \sqrt{\frac{mg}{I}} = \sqrt{\frac{3g}{h}}$$

where $I = mh^2/3$ is the moment of inertia of the uniform stiff rod rotating around its end. Finally, the total velocity of the head is

$$v_h = \sqrt{v'^2_{Lx} + 3gh} = 13 \text{ m/s}.$$ 

The vertical component is

$$v_{yv} = \sqrt{3gh} = 6 \text{ m/s}.$$ 

The head can sustain collisions at this speed (if one is wearing a helmet), but even at slow speed head trauma can lead to serious consequences, as the tragic of Natasha Richardson has shown.

4. (20 points) Loop-the-loop
You have seen this in-class experiment. A solid brass ball of mass $m = 70 \text{ g}$ rolls smoothly without slipping along a loop-the-loop track when released from rest at elevation $h$ along the straight section (see picture below). The circular loop has radius $R = 30 \text{ cm}$, and the ball has radius $r = 0.5 \text{ cm}$. Ignore rolling friction.

(a) (15 points) What is $h$ if the ball is on the verge of leaving the track when it reaches the top of the loop?

(b) (5 points) What is the magnitude and direction of the horizontal force component acting on the ball at point Q?

4. Solution
(a) At the top of the loop, two forces are acting on the ball: the normal force $N$ and the gravity force $mg$. Both are directed downward, and are providing centripetal acceleration of the ball:

$$N + mg = ma_{cp}$$

While the ball is in contact with the track, $N \geq 0$. The fact that it is on the verge of loosing contact means
\(N \approx 0\). Therefore, the centripetal force is generated by gravity only, and the centripetal acceleration is equal to \(g\):

\[a_{cp} = \frac{v_{cm}^2}{R} = g\]

where \(v_{cm}\) is the velocity of the center of mass at the top of the loop. Hence, the square of the velocity at the top is

\[v_{cm}^2 = gR \quad (9)\]

Now we need to relate that velocity to the initial elevation of the ball \(h\), and then solve for \(h\). We will use energy conservation for that. The initial energy of the ball at elevation \(h\) is its potential energy:

\[E_{\text{initial}} = mgh\]

where \(m\) is the mass of the ball. At the top of the loop, the energy of the ball is

\[E_{\text{top}} = \frac{mv_{cm}^2}{2} + I\omega^2 + mg(2R) \quad (10)\]

The first term is the kinetic energy of translation, the second is the kinetic energy of rotation, and the last part is the potential energy of the ball.

For smooth rolling, the center-of-mass velocity is related to the angular velocity of rotation:

\[v_{cm} = \omega R\]

Expressing \(\omega = v_{cm}/R\) and plugging in the moment of inertia for the solid sphere of mass \(m\) and radius \(r\) \((I = 2/5mr^2)\) into Eq. (10), we get

\[E_{\text{top}} = \frac{mv_{cm}^2}{2} \frac{7}{10} + 2mgR\]

For smooth rolling, friction is negligible, so the energy at the top is equal to the initial energy at elevation \(h\):

\[mgh = \frac{7}{10}mgR + 2mgR = \frac{27}{10}mgR\]

where we used in Eq. (9) to express \(v_{cm}^2 = gR\). Therefore,

\[h = 2.7R = 81 \text{ cm}\]

(b) Similar logic applies to point \(Q\). Here, the normal force is horizontal and nonzero, and it provides centripetal acceleration:

\[N_Q = ma_{cp} = \frac{mv_Q^2}{R}\]

We can find \(v_Q\) similar to part (a) from energy conservation. The energy at \(Q\) is

\[E_Q = \frac{7}{10}mv_Q^2 + mgR\]

where the first component is the sum of translational and rotational kinetic energies, and the second term is the potential energy at point \(Q\). Again, energy conservation requires

\[E_Q = E_{\text{initial}} = mgh\]

and

\[mv_Q^2 = \frac{10}{7}mg(h - R) = \frac{17}{7}mgR\]

(using \(h = 2.7R\) from part (a) ). Therefore, the horizontal force at point \(Q\) is

\[N_Q = \frac{17}{7}mg = 1.7 \text{ N}\]

5. (17 points) Grizzly Peak Rd

On my daily way to Berkeley, I usually take the Grizzly Peak Road, avoiding traffic in the Caldecott Tunnel. Grizzly Peak Rd is a narrow two-lane road, with some nice hairpin turns and stunning views of the Bay (can’t find a better start to a day ! But I digress). One of the turns on this road has a radius of about 10 meters.

(a) (5 points) Assuming the static coefficient of friction \(\mu = 1\) between my tires and the pavement, what is the maximum safe speed for this turn ?

(b) (12 points) A motorcyclist, rounding the curve at the same speed, will have to tilt the bike into the turn to maintain balance. What angle with the horizontal does he make?

5. Solution

(a) The first part of this problem is straightforward and was worked out in class. The car experiences three forces: gravity \(mg\) pointing downward, normal force \(N\)
pointing upward, and static friction $F_{fr}$ pointing toward the center of the turn. The balance of forces in the vertical direction implies

$$N = mg$$

The friction force provides centripetal acceleration:

$$F_{fr} = \frac{mv^2}{R}$$

where $m$ is the mass of the car, $v$ is the linear velocity, and $R = 10$ m is the radius of the turn. At the largest possible velocity, friction force reaches its maximum possible value

$$F_{fr}^\text{max} = \mu N = \mu mg$$

Therefore, the velocity has to be smaller than

$$v_\text{max} = \sqrt{\frac{F_{fr}^\text{max} R}{m}} = \sqrt{gR} = 10 \text{ m/s} = 22 \text{ mph}$$

(b) This part is a classic problem, and it describes a situation we all experience when we ride a bike, skate, etc. From experience, we know that in order to turn, the bike needs to be tilted into the turn. Why?

Let’s consider the process and its free-body diagram below.

There are three forces acting on the motorcycle: friction $F_{fr}$, normal force $N$ (both applied to the contact point with ground), and gravity $mg$ (applied to the center of mass point). Let’s assume that the distance between the center of mass point and ground is $h_{cm}$. The force balance (Newton’s law) equations are the same as in part (a):

$$N = mg; \quad F_{fr} = ma_{cp} = \frac{v^2}{R}$$

Now we need to write down the torque balance equations.

The choice of axis of rotation can greatly simplify this problem. The best choice here is the center of mass (CoM). Why? If we choose CoM as the axis of rotation, we do not have to worry about torque due to gravity, and we do not have to worry about $a_{cp}$, the acceleration of the center of mass. This is a general property of the solid bodies. You can always represent its motion as a superposition of the motion of the center of mass (in this case, with acceleration $a_{cp}$), and the rotation relative to the CoM. In the torque equation about the CoM, only real forces will appear.

If the bike is in equilibrium, the net torque about CoM has to be zero. The equation is

$$F_{fr} h_{cm} \sin \phi - Nh_{cm} \cos \phi = 0$$

which, together with $F_{fr} = \mu N$ means

$$\tan \phi = \frac{N}{F_{fr}} = \frac{1}{\mu} = 1$$

or $\phi = 45^\circ$.

You can also write the torque equation about the point of contact with ground, but then you have to take into account the fact that the center of mass is moving with acceleration $a_{cp}$. That means there is non-zero angular acceleration of the center of mass relative to the ground point, and that angular acceleration is created by the gravitational torque. The equation is

$$\tau = mgh_{cm} \cos \phi = (mh_{cm}^2)\alpha = (mh_{cm}^2) \frac{a_{cp} \sin \phi}{h_{cm}}$$

which simplifies to

$$\tan \phi = \frac{g}{a_{cp}}$$
Using $a_{cp} = F_{fr}/m = \mu g$, we get the same answer: $\phi = 45^\circ$.

Finally, you can work out this problem using fictitious centrifugal force. We generally do not advise this, since the fictitious forces have to be added “by hand”. But often they can greatly simplify the reasoning. In the reference frame of the bike, there is a fictitious centrifugal force, applied to the CoM point, and equal to $F_{cf} = -ma_{cp}$. This force applies an apparent torque on the body of

$$\tau_{cf} = F_{cf} h_{cm} \sin \phi$$

and this torque is balanced by the gravitational torque:

$$ma_{cp} h_{cm} \sin \phi = mg h_{cm} \cos \phi$$

which again leads to the same answer $\tan \phi = 1/\mu$.

The latter reasoning is in fact what you experience when you ride the bike. You have to tilt the bike into the turn in order to counteract the centrifugal torque, which is trying to push you in the direction away from the turn.