1. (10 points) Blitz
This is a set of simple questions to warm you up. The problem consists of five questions, 2 points each.

1. The distance between San Francisco and Honolulu is 2128 nautical miles. 1 nautical mile is equal to 1.852 km. A typical plane ride from San Francisco to Honolulu takes 5 hours. What is the average speed of the plane in m/s (to 1 significant digit) ?
Answer: \( v = \frac{d}{t} \approx 200 \text{ m/s} \).

2. Circle correct answer. A pitcher tosses a baseball straight up with an initial speed of 12 m/s. After reaching its maximum height, the ball falls back to the ground. Which of the following applies at the highest point of the trajectory ?
   (a) Only the instantaneous velocity is zero.
   (b) Only the acceleration is zero.
   (c) Both the instantaneous velocity and the acceleration are zero.
   (d) Neither the instantaneous velocity nor the acceleration is zero.
Answer: (c)

3. Circle correct answer. A projectile is launched at an upward angle of 30° to the horizontal with a speed of 30 m/s. How does the horizontal component of its velocity at \( t = 1.0 \text{ s} \) after launch compare with its initial horizontal component of velocity (at \( t = 0 \)), ignoring air resistance?
   (a) Value at \( t = 1.0 \text{ s} \) is larger than the value an \( t = 0 \)
   (b) Value at \( t = 1.0 \text{ s} \) is smaller than the value an \( t = 0 \)
   (c) Value at \( t = 1.0 \text{ s} \) is equal to the value an \( t = 0 \)
   (d) None of the above (explain)
Answer: (c)

4. Circle correct answer. When a vector sum of all forces acting on a body is zero, the body
   (a) moves with increasing speed
   (b) moves with decreasing speed
   (c) moves with constant speed, or remains at rest
   (d) All of the above
   (e) None of the above
Answer: (c)

5. Circle correct answer. Drag force acting on a moving object in air
   (a) Is always horizontal, and proportional to the normal force.
   (b) Is always vertical, and increases with velocity of the object.
   (c) Points in the direction opposite to the relative velocity in air, and increases with speed
   (d) Parallel to the velocity vector, and decreases with speed.
   (e) None of the above.
Answer: (c)

2. (25 points) Air travel
An airplane flies a straight course (with respect to the ground) from point \( P \) to point \( Q \) and then back to \( P \), with air speed (speed relative to air) which is always
equal to a constant value $U_0$, regardless of the wind. Wind of speed $W_0$ (relative to ground) is blowing at the angle $\theta = 45^\circ$ relative to the direction $\overrightarrow{PQ}$.

(a) (15 points) Find the time required for the round trip. Express your answer in terms of $U_0$, $W_0$, and the distance $D$ between $P$ and $Q$.

(b) (10 points) For which value of the wind speed $W_0$ the travel time is least?

2. Solution

(a) If the air velocity (velocity with respect to the air) of an airplane is $\mathbf{u}$, and the wind velocity with respect to the ground is $\mathbf{w}$, then the ground velocity $\mathbf{v}$ of the airplane is

$$\mathbf{v} = \mathbf{u} + \mathbf{w},$$

where $|\mathbf{u}| \equiv U_0$ is the plane’s airspeed, and $|\mathbf{w}| \equiv W_0$ is the speed of the air relative to ground. The idea of this problem is that the plane must oppose any perpendicular wind speed to maintain its straight path. Let the distance between $P$ and $Q$ be $D$.

Let the angle between the wind direction and $\overrightarrow{PQ}$ be $\theta = 45^\circ$. The perpendicular component of the wind speed is $w_\perp = W_0 \sin \theta = -u_\perp$. The airspeed in the parallel direction can be computed

$$u^2 = u_\perp^2 + u_\parallel^2 \Rightarrow u_\parallel = \sqrt{U_0^2 - W_0^2 \sin^2 \theta}$$

The wind also has a component along the direction of travel. This parallel component is $w_\parallel = W_0 \cos \theta$. On one leg of the trip (left vectors), this adds to the ground velocity. On the other leg (right vectors), it subtracts. This gives us the following formula:

$$T = D \left( \frac{1}{\sqrt{U_0^2 - W_0^2 \sin^2 \theta} + W_0 \cos \theta} + \frac{1}{\sqrt{U_0^2 - W_0^2 \sin^2 \theta} - W_0 \cos \theta} \right)$$

This can be simplified considerably:

$$T = \frac{2D \sqrt{U_0^2 - W_0^2 \sin^2 \theta}}{U_0^2 - W_0^2} = \frac{D \sqrt{2(U_0^2 - W_0^2)}}{U_0^2 - W_0^2}$$

(1)

(b) This part requires some calculus. We need to do a minimization of a function. What this part asks is to study the travel time as a function of wind speed $W_0$, given by Eq. (1). Remember that functions have maxima and minima at places where the derivative vanishes, so we need to take the derivative of $T$ with respect to $W_0$:

$$\frac{d}{dW_0} T(W_0) = 2D \left( \frac{2W_0 \sqrt{U_0^2 - W_0^2 \sin^2 \theta}}{(U_0^2 - W_0^2)^{3/2}} - \frac{W_0 \sin^2 \theta}{\sqrt{U_0^2 - W_0^2 \sin^2 \theta} (U_0^2 - W_0^2)^{3/2}} \right)$$

The derivative is clearly zero when $W_0 = 0$ for any value of $\theta$. In other words, the round trip is always fastest with no wind! In this case the travel time $T = D/U_0$.

There is another case we have to worry about though. We can divide out unimportant pieces to get an equation for another value where the derivative vanishes

$$U_0^2 \sin^2 \theta - W_0^2 \sin^2 \theta = 2U_0^2 - 2W_0^2 \sin^2 \theta$$

This gives us the other point where the derivative is zero

$$W_0^2 = U_0^2 \frac{2 - \sin^2 \theta}{\sin^2 \theta}$$

Notice that this point always occurs when the wind speed is greater than the air speed. No progress can be made against the wind if this is the case, so the trip cannot occur. The final possibility to consider is the case where the wind speed is exactly the same as the air speed. Looking Eq. (1), the time taken is infinite. So the only possible answer is that the minimum is at $W_0 = 0$.

3. (20 points) Kickoff at the Memorial Stadium

Giorgio Tavecchio is kicking off against Michigan State for the Golden Bears. The ball leaves his foot at ground level with a typical velocity of $V_0 = 23$ m/s. For this problem, we will ignore air resistance and assume there is no wind (Cal fans are holding their collective breath).

• (5 points) Determine the maximum possible distance of the kick. Note that the angle of the kick
is not given; you have to determine what the optimal angle for the kick would be, and assume that Mr. Tavecchio has done the same. Provide relevant explanations.

- \((10 \text{ points})\) Rather than kicking for distance, it is often important to kick the ball in such a way so that a player from your team (the gunner) arrives at the spot where the ball lands exactly at the time it lands. Imagine that Jahvid Best is playing the gunner, and starts running downfield with the speed of \(U = 10 \text{ m/s}\), starting at the time the ball is kicked from the same point. At what angle over the horizon should Mr. Tavecchio kick, so that the ball and the gunner arrive at the same spot downfield at the same time? What is the length of the kick (total distance traveled by the ball along horizontal direction)?

- \((5 \text{ points})\) How valid is the assumption that air resistance is negligible? Make quantitative arguments, assuming the mass of the regulation football is \(m = 0.43 \text{ kg}\), its typical cross section area is \(0.03 \text{ m}^2\), and the drag coefficient is \(C_D \approx 1\). The density of air is \(\rho = 1.3 \text{ kg/m}^3\).

3. Solution

(a) We have derived the expression for the maximum distance in class:

\[ L = \frac{V_0^2 \sin 2\alpha}{g} \]  

(2)

where \(\alpha\) is the angle the initial velocity vector makes with the horizon. From Eq. (2), the horizontal distance \(L\) is maximized when \(\sin 2\alpha = 1\), \(i.e.\ \alpha = 45^\circ\). Therefore,

\[ L_{\text{max}} = \frac{V_0^2}{g} = 54 \text{ m}. \]

However, to get the full credit, you must at least explain why Eq. (2) is applicable. We derived it for the following conditions:

1. Acceleration is constant and is equal to \(a = (0, -g)\). In other words, no air resistance, and no wind.

2. The kick starts and ends at the same elevation (which is the case here, as the ball is kicked from ground level and lands at ground level).

(b) This situation is conceptually similar to the “Shoot the Monkey” demo in class. Two objects (the ball and the gunner) are moving simultaneously, starting and the same initial position and at the same time. The equations for the ball are

\[ x_b = V_0 t \cos \alpha \]  

(3)

\[ y_b = V_0 t \sin \alpha - \frac{gt^2}{2} \]  

(4)

and the gunner’s motion is purely horizontal:

\[ x_g = Ut. \]  

(6)

We require that at the time \(t = T\) when the ball lands \((y_b(T) = 0)\), the positions of the ball and the gunner are the same:

\[ x_b(T) = x_g(T). \]

The latter condition determines the optimal angle; combining Eq. (3) and Eq. (6):

\[ V_0 T \cos \alpha = UT \Rightarrow \cos \alpha = \frac{U}{V_0} \]  

(7)

and \(\alpha = 64^\circ\). Eq. (7) is obvious: since you can replace a specific time \(T\) with any time \(t\), it means that the ball and the gunner are always at the same \(X\) position, \(i.e.\) the ball is moving perfectly vertically relative to the gunner. This is very similar to the “ballistic cart” demo I did in lecture, where a ball was launched up from a moving cart.

With this angle \(\alpha = 64^\circ\), the horizontal range of the kick is still given by Eq. (2):

\[ L = \frac{V_0^2}{g} \sin 2\alpha = 42 \text{ m}. \]

(e) At these speeds, the air motion around the ball is turbulent, and the drag is proportional to the square of the velocity:

\[ D = \frac{1}{2} C_D \rho A v^2 \]

where \(C_D \approx 1\) is the drag coefficient (appropriate for a ball tumbling end-over-end, as often happens for kicked
balls), $A \approx 0.03 \text{ m}^2$ is the cross sectional area, and $v$ is the airspeed. The acceleration caused by the drag force early in the trajectory (when $v \approx V_0$) is therefore

$$a_D = \frac{C_D \rho AV_0^2}{2m} \approx 24 \text{ m/s}^2$$

or more than twice the acceleration of gravity! That clearly cannot be ignored. Even if the drag coefficient is smaller by an order of magnitude (for a perfectly-thrown ball spiraling along its major axis $C_D \approx 0.1$), the drag is large enough that the usual approximation of “ignoring air resistance” causes you to significantly overestimate the range of the ball [1]. Now you can understand why the kickers like to kick in Denver, where the air density is significantly (up to 20%) smaller than at sea level!

4. (20 points) Crossing a river

A group of hikers is using a rope trolley to cross a river. One of them attaches one end of the rope to a tree on the far bank of the river. The rest of the group, after securing the other end of the rope tightly to a tree on the near bank, pull themselves across the river one by one, while being suspended on the rope with a harness. The distance between the trees is $L = 20$ m. See picture below.

(a) (10 points) When the heaviest of the hikers, a 100 kg gentleman, is exactly midway between the trees, the rope sags by $H = 1$ m. What is the tension $T$ of the rope?

![Diagram](https://via.placeholder.com/150)

(b) (10 points) Hopefully, you have found out that the tension $T$ is large. Clearly, a few people would have a hard time stretching the rope to such tension. So it is common to use a block-and-tackle device to pull the rope until tension reaches the required value $T$. Block-and-tackle (remember lecture demonstration?) is a set of pulleys, arranged as schematically shown below (two pulleys on the left and one end of the tackle rope are connected to the tree, the other two pulleys are connected to the rope across the river). Suppose a single hiker wants to be able to tighten the rope across the river to tension $T$ found in part (a), by pulling with force $F = 500$ N ($\approx 110$ lbs). How many pulleys (total, on both sides) does he need in the block-and-tackle device? (If you did not get the answer in part (a), express your answer to part (b) symbolically, in terms of rope tension $T$).

4. Solution

(a) The force diagram for the hiker is shown below.

![Force Diagram](https://via.placeholder.com/150)

In equilibrium,

$$T_1 + T_2 + m\vec{g} = 0$$

The tension forces forces are equal in magnitude $|T_1| = |T_2| \equiv T$ and balance out in the horizontal direction. In the vertical direction, the force equation is

$$|T_1| \sin \theta + |T_2| \sin \theta - mg = 0$$

where $m = 100$ kg is the mass of the hiker. This yields, for the magnitude of tension $T$

$$T = \frac{mg}{2 \sin \theta}$$

The angle $\theta$ can be found from right triangle formed by two sides $H$ and $L/2$:

$$\tan \theta = \frac{2H}{L} = 0.1 \approx \sin \theta$$

The last (approximate) equality is good to 0.5%. Plugging the numbers in, the magnitude of the tension of the rope is

$$T = \frac{mgL}{4H} = 5 \text{kN}.$$
(b) The difference in length between the straight rope and the rope deflected by $H = 1 \text{ m}$ is only 0.5%, so the tension in a straight rope is also approximately $T = 5 \text{ kN}$, or about half a ton (that was explicitly stated in part (b)). A single person can’t pull the rope that tightly without a mechanical advantage device. So a block-and-tackle (either a dedicated pulley device, or a jig rigged with carabiners) is often handy. The tension needs to be high so that the rope trolley bridge does not deflect too much under the weight of a person (see part (a)): trying to pull oneself along the rope up the steep incline is not fun! Incidentally, 5 kN is not too much tension for good rope: a 1 cm diameter high-quality braided rope could often hold up to 20 tons.

We discussed the block-and-tackle in class. When the block-and-tackle is hooked up to the rope as shown, the mechanical advantage is $N$, the number of pulleys in the system (or twice the number of pulley pairs. That is, the movable pulleys are connected directly to the rope being tightened. If we pull with zero acceleration, the tension of the tackle (the thread between the pulleys) is equal to $F$, the force applied by the person. At the same time, the force applied to the rope is $2F$ per movable pulley, since the tension force of the tackle on each side of each movable pulley is directed to the left. To provide the total force $T$ on the rope, you need to pull with force

$$F = \frac{T}{N},$$

therefore, the number of pulleys is

$$N = \frac{T}{F} = 10.$$

5. (25 points) Ski Jump

Ski jumping is a nordic sport featured in the Winter Olympics. Imagine an athlete sliding down a 45° ski ramp of height $H = 140 \text{ m}$, starting at rest. The ramp flattens out to horizontal direction at elevation $h = 70 \text{ m}$, where the skier takes off (see picture). The coefficient of friction between the skis and the ramp is $\mu = 0.1$. You can ignore the length of the flat section at the bottom of the ramp, as well as the air resistance and any lifting force in the air.

(a) (10 points) What is the speed of the skier at the bottom of the ramp (just before takeoff) ?

(b) (10 points) Find the horizontal flight distance $s$ and flight time.

(c) (5 points) In the Olympics, the typical jumps can be as long as 200 m. If your calculation in part (b) deviates significantly from this distance, what assumption we made was likely incorrect? Explain.

5. Solution

(a) First, we will find the launch velocity using acceleration. To find acceleration, we need to compute the net force acting on the skier. We start by drawing the free-body diagram

The normal force $N$ is balanced in the $Y$ direction (normal to the ramp) by the projection of the gravity:

$$N = mg \cos \alpha$$

which means the friction force is

$$F_{fr} = \mu N = \mu mg \cos \alpha$$

In $X$ direction (along the ramp),

$$a_x = \frac{mg \sin \alpha - F_{fr}}{m} = g(\sin \alpha - \mu \cos \alpha)$$

Using the standard kinematic formula

$$v_f^2 - v_i^2 = 2a_x \Delta x$$
and \( v_f = v, v_i = 0 \), \( \Delta x = L = (H - h)/\sin \alpha \), we get

\[
v^2 = 2g(H - h)\frac{\sin \alpha - \mu \cos \alpha}{\sin \alpha}
\]

\[
v = \sqrt{2g(H - h)\left(1 - \frac{\mu}{\tan \alpha}\right)} = 35 \text{ m/s} \quad (8)
\]

We can also find the final velocity of the skier at the bottom of the ramp using work-energy theorem that we just discussed in class (this was not required for the exam, but some of you may be curious). The initial energy at the top of the ramp is purely potential

\[
E_{\text{top}} = mgH
\]

and the final energy is a combination of potential and kinetic:

\[
E_{\text{bottom}} = mgh + \frac{mv^2}{2}
\]

Since there is a friction force on the ramp, the total mechanical energy is not conserved, but instead the change in energy is the work of the friction force:

\[
E_{\text{bottom}} - E_{\text{top}} = W_{\text{fr}} = -F_{\text{fr}}L
\]

where \( F_{\text{fr}} = \mu N \) is the magnitude of the friction force, and \( L = (H - h)/\sin \alpha \) is the length of the ramp. Putting everything together, we can solve for velocity at the bottom:

\[
\frac{mv^2}{2} + mgh - mgH = -\mu mgL
\]

\[
v = \sqrt{2g(H - h)\left(1 - \frac{\mu}{\tan \alpha}\right)} = 35 \text{ m/s} \quad (9)
\]

which is identical to Eq. (9).

(b) Now we will use the projectile motion. The initial velocity of the skier is horizontal. The horizontal distance he covers is given by

\[
s = vt_{\text{flight}}
\]

where \( t_{\text{flight}} \) is the time he is in the air. Since his initial vertical velocity is zero, the elevation \( h \) is related to the flight time:

\[
h = \frac{gt_{\text{flight}}^2}{2}
\]

We solve this first for time:

\[
t_{\text{flight}} = \sqrt{\frac{2h}{g}} = 3.8 \text{ sec}
\]

Then, plugging into Eq. (10), we find the flight distance

\[
s = vt_{\text{flight}} = 133 \text{ m}
\]

(c) In fact, the Olympic skiers routinely record jumps of over 200 m, and they land at relatively modest speeds. The reason for this discrepancy is air resistance. In horizontal direction, it slows the skier down, so naively it would decrease the range. But a more important effect happens in the vertical direction, where air resistance provides a lift. In other words, air slows down skier’s fall, making the flight longer, and reducing the landing speed. In fact, if you watch this even in the Olympics, you may notice that the jumpers orient their skis and the bodies so that they act like wings. The Flying Finns are pretty good at this!