1. (15 points) Race Strategy

Two swimmers need to get from point A on one bank of the river to point B directly across the river on the opposite bank. One of them decides to swim along the line AB. The other swims perpendicular to the current to the opposite bank, and then walks with velocity $u$ upstream to B from the point where he gets out of the water. Find the value of $u$ such that both swimmers reach B at the same time. Assume that the speed of each swimmer relative to the water is $v' = 2.5 \text{ km/h}$, and the speed of the water is $v_0 = 2.0 \text{ km/h}$.

1. Solution

$$
\begin{align*}
&V_0 \quad V_1 \quad V_2 \\
&V'_0 \quad V'_1 \quad V'_2 \\
&A \quad B \quad C
\end{align*}
$$

This is a problem about relative velocities and addition of vectors. The first swimmer moves along line AB. His velocity, relative to the bank, is a vector sum

$$v_1 = v'_1 + v_0$$

and its magnitude, according to the picture above is

$$v_1 = \sqrt{v'^2 - v_0^2}$$

The amount of time it takes to cross the river is

$$t_1 = \frac{|AB|}{v_1} = \frac{|AB|}{\sqrt{v'^2 - v_0^2}}.$$ 

The second swimmer crosses the river with velocity

$$v_2 = v'_2 + v_0$$

relative to the bank. Its component in the direction of AB is always $v'$, so it takes him

$$t_{2_{\text{swim}}} = \frac{|AB|}{v'}$$

to cross the river. After that, he needs to walk from C to B, which takes

$$t_{2_{\text{walk}}} = \frac{|BC|}{u}$$

The total time is, therefore,

$$t_2 = \frac{|AB|}{v'} + \frac{|BC|}{u} \quad (1)$$

From geometry, it is clear that

$$|BC| = |AB| \frac{v_0}{v'}.$$

Inserting that into Eq. (1) and requiring $t_1 = t_2$ we get an equation for $u$:

$$|AB| \left( \frac{1}{v'} + \frac{v_0}{uv'} \right) = \frac{|AB|}{\sqrt{v'^2 - v_0^2}}$$

and after some algebra

$$u = v_0 \frac{\sqrt{v'^2 - v_0^2}}{v' - \sqrt{v'^2 - v_0^2}} = 3.0 \text{ m/s}.$$ 

2. (20 points) John Kruger and his parachute

There is a scene in the movie Eraser, where the hero John Kruger (Arnold Schwarzenegger) is falling head first from an airplane, trying to catch up with a folded parachute. Let’s see if Hollywood got some basic physics right.

In the scene, Kruger drops the parachute first. We will assume, that the parachute is falling with the constant vertical velocity $v_p = 20 \text{ m/s}$ (that’s roughly the terminal velocity for a folded parachute). $\Delta t = 7 \text{ seconds}$ later, after dodging a few bullets from the bad guys, Kruger lets go of the plane, and falls head first after the parachute. We will assume that the drag force acting on John is negligible.
(a) (10 points) At what altitude, relative to the plane, does John catch up with the parachute?

(b) (5 points) What is his velocity at that point?

(c) (5 points) If he continues to fall for another 20 seconds, while trying to put on the parachute, how much farther does he fall? Again, assume that drag is negligible.

(d) (Bonus 5 points) Do you see anything wrong with this scenario?

2. Solution

(a) Let’s start the clock at \( t_0 = 0 \) when John starts falling and use \( y_0 = 0 \) to be the altitude of the plane. Then, the parachute starts falling at \( t_p = -\Delta t \). Let’s also follow the tradition and point the \( Y \) axis up (so that negative altitude means, as usual, that the object is below the plane). You may have chosen to point your \( Y \) axis downward, in which case all positions and velocities below would switch sign. The equations of motion for John and the parachute are

\[
y_J(t) = \frac{-gt^2}{2} \\
y_p(t) = -v_p(t + \Delta t)
\]

(2, 3)

(notice signs). We want to find such time \( t_c \) so that \( y_J(t_c) = y_p(t_c) \). The equation for \( t_c \) follows from Eqs. (2, 3):

\[
\frac{g}{2}t_c^2 - v_p t_c - v_p \Delta t = 0
\]

which has one allowed (positive) solution

\[
t_c = \frac{v_p + \sqrt{v_p^2 + 2v_p g \Delta t}}{g} = 7.7 \text{ s}.
\]

At the instance \( t = t_c \), John would be at the elevation

\[
y_J(t_c) = \frac{gt_c^2}{2} = -290 \text{ m}.
\]

(b)

\[
v_J(t_c) = -gt_c = -77 \text{ m/s}.
\]

This velocity is high, but even at 80 m/s, the drag force acting on John is not significant, if he is falling headfirst. His cross sectional area is about \( A \approx 0.04 \text{ m}^2 \), and the drag force is approximately

\[
F_d \approx 0.25 (\text{kg}/\text{m}^3)Av_c^2 = 60 \text{ N}
\]

or about 5% of Schwarzenegger’s weight. So ignoring drag for Kruger is justified.

(c) Between \( t = t_c \) and \( t_2 = t_c + 20 \text{ sec} \), John and his parachute have fallen by

\[
\Delta y = -v_J(t_c)(t_2 - t_c) - \frac{g}{2}(t_2 - t_c)^2 = -3.5 \text{ km}
\]

So John’s final altitude at that time \( t_2 \) would be

\[
y_2 = y_J(t_c) + \Delta y = -\frac{gt_c^2}{2} = -3.8 \text{ km}.
\]

Now, how realistic is that? First off, in the movie John spent about 20 sec (instead of 8) to catch the parachute. I guess that adds to the dramatics, but it just can’t be right, unless someone added a lead brick to John’s parachute bag (which would make \( v_p \) significantly larger). Also, in 20 sec, John would have fallen by 2 km – and would likely have created a significant crater on the ground.

The total fall distance of almost 4 km is also problematic. In the movie, the pilot can be heard saying “Descending to 3000 feet” 20 sec before John jumps. Indeed, when John opens the door, there is no significant pressure difference between the plane and the outside, the oxygen masks don’t appear, etc. This means the plane was probably flying below 8000 feet, which is only 2.4 km. Yet, when John opens the parachute, he appears to still be fairly high in the air – at least 3000 feet or so.

The final goof that amused me is the motion along plane’s path (which we ignore for this problem, but have to account for in real life). Indeed, in 7 seconds, the jet, which should have had a cruising speed of at least 200 km/h, would have traveled by about 400 meters. The parachute, on the other hand, would quickly reach the horizontal velocity of \( v_p \) (it can be seen flying backwards relative to the plane). When John jumps, he still has a horizontal velocity along the plane’s direction – so he is flying away from the chute. How can he ever catch up? Schwarzenegger may be the Eraser, but he’s no Superman...

3. (25 points) Ski Jumper

Ski jumping is a nordic sport to be featured in the upcoming Winter Olympics next year. Imagine an athlete sliding down a ski jump of height \( H \), starting at rest. Consider the jump to be a ramp with the constant incline until it becomes horizontal near the release point.
Ignoring friction, what elevation of the jump point \( h \) would provide the maximum flight distance \( s_{\text{max}} \), and what is that distance?

3. Solution

First, let's express distance \( s \) in terms of parameters \( H \) and \( h \), and then find when \( s \) is maximum. Suppose velocity of the skier at the release point is \( v \). The velocity is horizontal. After the release, the skier will move under the influence of gravity, and his \( x \) and \( y \) coordinates will depend on time as

\[
x(t) = vt \\
y(t) = h - \frac{gt^2}{2}
\]  

Plugging in \( x(t) = s \) and \( y(t) = 0 \) (condition for landing), we get

\[
s = v \sqrt{\frac{2h}{g}}. \tag{6}
\]

Now we need to find the velocity \( v \) at the release point. Let's consider the motion of the skier along the ramp. First, we draw the free-body diagram:

In the \( Y \) direction, the normal force is balanced by the \( Y \) projection of the gravity:

\[ N = mg \cos \alpha \]

In the \( X \) direction, the acceleration is created by the unbalanced \( X \) projection of the gravity force:

\[ ma_x = mg \sin \alpha \Rightarrow a_x = g \sin \alpha \]

If the length of the ramp is \( S \), the velocity of the skier at the bottom of the ramp is related to acceleration through

\[
\frac{v^2}{2} = a_x S
\]

(which is one of the kinematic formulae derived in class), which means

\[
v = \sqrt{2a_x S} = \sqrt{2gS \sin \alpha} = \sqrt{2g(H - h)} \tag{7}
\]

where we used \( H - h = S \sin \alpha \) from the properties of the right triangle.

Plugging the expression from Eq. (7) into Eq. (6), we find

\[
s = 2 \sqrt{h(H - h)}. \tag{8}
\]

In order to find the maximum distance, we need to require that the first derivative of \( s \) w.r.t. to \( h \) (the only variable in the problem) is zero:

\[
\frac{ds}{dh} = \frac{H - 2h}{\sqrt{h(H - h)}} = 0
\]

The solution is

\[
h = H/2; \ s_{\text{max}} = H.
\]

Notice that Eq. (7) follows very straightforwardly from the work-energy theorem we discussed in class this week. Even though you are not expected to use the work-energy theorem on the exam, it is an allowed tool, if you are comfortable with it.

4. (40 points) Three Blocks

What should the horizontal acceleration \( a \) of block \( A \) be such that small blocks 1 and 2 are not moving relative to it? Masses of blocks 1 and 2 are equal, and the coefficient of friction between them and block \( A \) is equal to \( \mu \). Ignore pulley and string masses and friction in the pulley.
4. Solution

This problem would be much too involved for the actual exam, but it is, nonetheless, an excellent practice example.

The trick in this problem is to realize where the normal force that creates friction between block 2 and block A comes from and to set up force equations correctly. We also should not be concerned with forces acting on block A too much – it is moving with acceleration a – but it is irrelevant what force is causing this acceleration.

It is also important to draw the force diagrams correctly. It actually turns out to be quite relevant: the answer changes depending on the direction of the friction forces. What happens if we increase acceleration a beyond \( a = g \frac{1-\mu}{1+\mu} \)? At some point block 1 would start sliding to the left relative to A. At that point, the direction of the friction force \( F_1 \) should clearly be to the right. How about \( F_2 \)? If block 1 wants to slide left, block 2 will be pulled upwards – and \( F_2 \) should be directed downwards. Let’s rewrite our force equations for these directions and consider acceleration a such that the blocks 1 and 2 are still at rest relative to A, but just barely.

(12) \( \text{Block 1, } X \text{ projection} : ma = T + F_1 \)

(13) \( \text{Block 1, } Y \text{ projection} : 0 = N - mg \)

(14) \( \text{Block 2, } X \text{ projection} : ma = N_2 \)

(15) \( \text{Block 2, } Y \text{ projection} : 0 = T - mg - F_2 \)

You may notice that Eq. (12) and Eq. (15) are similar to Eq. (8) and Eq. (11) if we substitute \( \mu \rightarrow -\mu \). Therefore, the solution in this case is

\[ a = g \frac{1+\mu}{1-\mu} \]

If you’d like, you can show this explicitly using Eqs. (12-15). What will happen if acceleration \( a \) is between the two extremes, \( a = g \frac{1-\mu}{1+\mu} \) and \( a = g \frac{1+\mu}{1-\mu} \)? The blocks will still be at rest relative to A, but the friction forces would not be maximum (in other words, \( F_1 \leq \mu N_1 \) and \( F_2 \leq \mu N_2 \)). In particular, for \( a = g \), both friction forces are zero. \( F_1 \) points left and \( F_2 \) points up for

\[ g \frac{1-\mu}{1+\mu} \leq a < g \]

and \( F_1 \) points right and \( F_2 \) points down for

\[ g < a \leq g \frac{1+\mu}{1-\mu} \]

Notice that both forces change their direction at the same value of \( a = g \). In other words, you will never find
a situation where $F_1$ points left and $F_2$ points down (or the other way around, $F_1$ points right and $F_2$ points up). If that were possible, it would mean that in the absence of friction, blocks would be moving either towards or away from each other, making tension force either zero or infinite. This is not physical.