Systematic Errors

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“Another Class of Errors”

- **Statistical errors:**
  - Spread in values one would see if the experiment were repeated multiple times
    - RMS of the estimator for an ensemble of experiments done under the same conditions (e.g. numbers of events)
  - Several methods discussed
    - $(\text{sqrt})$variance of the estimator if PDF is known
    - Curvature of log(likelihood)
    - $\Delta \log(L) = \frac{1}{2}$ rule (or $\Delta \chi^2 = 1$)

- **But there is another source of uncertainty in results:** systematics
Simple Example

- **Mass spectrometer**

  \[ m = \frac{qrB^2}{2V} \]

- **Stat error: resolution/sqrt(N)**
  - Measure V,B for each run
  - Average fluctuations

- **Common errors do not average out**
  - Scale of B,V
  - Radius r
  - Velocity selection
  - Energy loss (residual pressure)
  - Etc, etc.
Combination of Errors

• Normally, independent errors are added in quadrature

  For instance, if measurements of \( r, V, B \) are uncorrelated, then (to first order)

\[
\frac{\sigma(m)}{m} = \sqrt{\left( \frac{\sigma(r)}{r} \right)^2 + \left( \frac{\sigma(V)}{V} \right)^2 + \left( 2 \frac{\sigma(B)}{B} \right)^2}
\]

• This is fine for a single ion

  But when we average (take more data), have to take into account the fact that errors on \( r, V, B \) correlate measurements of mass for each ion
Quadrature Sum

- Stat and syst errors are typically quoted separately in experimental papers (though not in PDG)
  - E.g. $\sigma = [15 \pm 5 \text{ (stat.)} \pm 1 \text{ (syst.)}] \text{ nb}$
  - It is understood that the first number scales with the number of events while the second may not
    - Splitting like this gives a feeling of how much a measurement could be improved with more data
    - It is also understood that stat and syst errors are uncorrelated (if this is not the case, have to say so explicitly !)
    - It is also understood that stat errors are uncorrelated between different experiments, while syst errors could be correlated (modeling, bias)
Classic Example (one of many)
Combining Errors

• For one measurement with stat and syst errors, this is easy

  - Suppose we measure \( x_1 = \langle x_1 \rangle \pm \sigma_1 \pm S \)
    - Split into “random” and “systematic” parts
      - \( x_1 = \langle x_1 \rangle + x^R + x^S \)
      - \( \langle x^R \rangle = \langle x^S \rangle = 0, \langle (x^R)^2 \rangle = \sigma_1, \langle (x^S)^2 \rangle = S \)
      - Total variance \( V[x_1] = \langle x_1^2 \rangle - \langle x_1 \rangle^2 = \langle (x^R + x^S)^2 \rangle = \sigma_1^2 + S^2 \)
      - Syst and stat errors are combined in quadrature
Combining Two Measurements

• Imagine two experiments, which measure $x_1$ and $x_2$ with common systematics $S$

  - Covariance:

    \[ Cov(x_1, x_2) = < x_1 x_2 > - < x_1 >< x_2 > \]
    \[ = < (x_1^R + x_1^S)(x_2^R + x_2^S) > - < x_1^R + x_1^S >< x_2^R + x_2^S >. \]
    \[ Cov(x_1, x_2) = Cov(x_1^S, x_2^S) = S^2. \]

  - Covariance matrix:

    \[ \mathbf{V} = \begin{pmatrix} \sigma_1^2 + S^2 & S^2 \\ S^2 & \sigma_2^2 + S^2 \end{pmatrix} \]
Generalization

• Imagine now 3 experiments: \( x_1, x_2, x_3 \)
  
  - All share common systematic error \( S \)
  - \( x_1 \) and \( x_2 \) share another error \( T \), but \( x_3 \) does not

\[
V = \begin{pmatrix}
\sigma_1^2 + S^2 + T^2 & S^2 + T^2 & S^2 \\
S^2 + T^2 & \sigma_2^2 + S^2 + T^2 & S^2 \\
S^2 & S^2 & \sigma_3^2 + S^2
\end{pmatrix}.
\]

Very few assumptions here: this works for non-gaussian distributions with large variances!
Error Propagation

• Fully fledged formula
  \[ V(f) = \left( \frac{df}{dx} \right)^2 V(x) + \left( \frac{df}{dy} \right)^2 V(y) + 2 \left( \frac{df}{dx} \right) \left( \frac{df}{dy} \right) Cov(x, y) \]

□ Consequences
  
  \( \text{If two measurements are correlated, it may be possible to find a combination with zero variance (det}(V)=0) \)
  
  \( \text{Two fully-correlated measurements } x_1, x_2; \text{ for } X=x_1+x_2: \)
  \[ V(X) = V(x_1) + 2Cov(x_1, x_2) + V(x_2) \]
  \[ = \sigma_1^2 + 2\sigma_1\sigma_2 + \sigma_2^2 \]
  \[ = (\sigma_1 + \sigma_2)^2 \]
  Errors add up linearly!
m functions $f_1, \ldots, f_m$, n variables $x_1, \ldots, x_n$:

$$\text{Cov}(f_k, f_l) = \sum_i \sum_j \left( \frac{\partial f_k}{\partial x_i} \right) \left( \frac{\partial f_l}{\partial x_j} \right) \text{Cov}(x_i, x_j)$$

Or in matrix form

$$G_{ki} = \left( \frac{\partial f_k}{\partial x_i} \right)$$

and

$$V_f = G V_x \tilde{G}$$
Example (Barlow)

Measure transverse momentum in axial magnetic field through $p_T=0.3B\rho$. For 2 particles, compute covariance matrix of measured momenta:

Example 5: With

$$f = \begin{pmatrix} P_{T1} \\ P_{T2} \end{pmatrix} \quad x = \begin{pmatrix} \rho_1 \\ \rho_2 \\ B \end{pmatrix}$$

Equation 1 gives

$$G = \begin{pmatrix} 0.3B & 0 & 0.3\rho_1 \\ 0 & 0.3B & 0.3\rho_2 \end{pmatrix}$$

and applying Equation 7 to

$$V_x = \begin{pmatrix} \sigma_{\rho_1}^2 & 0 & 0 \\ 0 & \sigma_{\rho_2}^2 & 0 \\ 0 & 0 & \sigma_B^2 \end{pmatrix}$$

gives the error matrix for the two $P_T$ measurements as

$$\begin{pmatrix} 0.3^2B^2\sigma_{\rho_1}^2 + 0.3^2\rho_1^2\sigma_B^2 & 0.3^2\rho_1\rho_2\sigma_B^2 \\ 0.3^2\rho_1\rho_2\sigma_B^2 & 0.3^2\rho_2^2\sigma_B^2 \end{pmatrix}$$
Systematic Errors and Fitting

- Use covariance matrix in $\chi^2$:

$$\chi^2 = \sum_i \sum_j d_i V_{ij}^{-1} d_j$$

$\downarrow$ $d_i=(y_i-y_i^{\text{fit}})$

$\downarrow$ Can apply the same recipe for ML fit (e.g. $L \sim \exp(-\chi^2/2)$)
**Example 8:** Consider a straight line $y = mx + c$ where all the $y$ values have a random error $\sigma$ and share a common systematic error $S$.

$$m = \frac{\overline{xy} - \overline{x} \overline{y}}{\overline{x^2} - \overline{x}^2} \quad c = \frac{\overline{x^2} \overline{y} - \overline{x} \overline{xy}}{\overline{x^2} - \overline{x}^2}$$

The full error formula gives, for the slope ($\Delta = \overline{x^2} - \overline{x}^2$)

$$V(m) = \frac{1}{N^2 \Delta^2} \sum_i \sum_j (x_i - \overline{x})(x_j - \overline{x})Cov(y_i, y_j)$$

And we have $Cov(y_i, y_j) = \delta_{ij} \sigma^2 + S^2$ so

$$V(m) = \frac{1}{N^2 \Delta^2} \sum_i (x_i - \overline{x})^2 \sigma^2 + \sum_i \sum_j (x_i - \overline{x})(x_j - \overline{x})S^2$$

The first summation gives the usual answer $\sigma^2/N\Delta$. The second gives zero, as $\sum x_i = \overline{x}$. Now for the constant,

$$V(c) = \frac{1}{N^2 \Delta^2} \sum_i \sum_j (\overline{x^2} - \overline{x}x_i)(\overline{x^2} - \overline{x}x_j)Cov(y_i, y_j)$$

The $\sigma^2$ values along the diagonal give the standard term $\sigma^2 \overline{x^2}/N\Delta$, but the other term does not cancel. $\sum(\overline{x^2} - \overline{x}x_i)$ is just $N\Delta$, giving an additional term $(N\Delta)^2S^2/(N^2\Delta^2)$ which is just $S^2$. The presence of an additional systematic (constant) error $S$ on the $y$ measurements does not affect the slope, but adds (in quadrature) to the uncertainty on the constant by an amount $S$. 
Practical Implications

• In the full formalism, can still use $\chi^2/\text{df}$ test to determine the goodness of fit
  ✔ But this will not work unless correlations are taken into account
  ✔ For simplicity, if all stat errors are roughly equal and all systematic errors are common, can do the fit with stat errors only (this will determine stat errors on parameters), then propagate syst errors

• Limitations
  ❐ More points do not improve the systematic error
  ❐ Goodness of fit would not reveal unsuspected sources of systematics
    ✔ All points move together -- same goodness of fit