Problem of the week: Oscillators

(a) (15 points) Imagine a straight tunnel bored through the center of the Earth from one point on the surface (say, near Berkeley, CA) to the other side (alas, this would be in the Indian Ocean somewhere!). We'll assume that the tunnel is carefully built to withstand large temperatures and pressures in the middle of the Earth, and it is evacuated well enough to ignore air resistance inside the tunnel. If you drop a rock into this tunnel, it will experience a gravity force directed straight toward the center of the Earth. Calculations show that the weight of the rock inside the tunnel depends linearly on the distance r from the center of the Earth: W(r)=mgr/R where m is the mass of the rock, $g = 9.8m/s^2$, and R=6400 km is the radius of the Earth. Prove that the motion of the rock inside the tunnel is periodic, and find the period.

SOLUTION: The gravitational force acting on the mass m is given by

$$F(r) = -\frac{mg}{R}r \tag{1}$$

where $g = 9.8m/s^2$ and $R = 6,400km = 6.4 \times 10^6 m$ is the radius of the Earth. The negative sign in (1) accounts for the direction of the gravity which is toward the center of the Earth. Note that the gravitational force, $F(r)$, is the only force acting on the mass. Therefore, the Newton's law for this rock becomes

$$F_{net} = ma = -\frac{mg}{R}r \tag{2}$$

where a is the resultant acceleration and we can rewrite it as

$$ma = m\frac{d^2r}{dt^2} = m\ddot{r} = -\frac{mg}{R}r. \tag{3}$$

Then, move the term $-\frac{mg}{R}r$ on LHS of (3) to RHS to get the equation of oscillation (please look up the oscillation chapter of the textbook),

$$\ddot{r} + \frac{g}{R}r = 0. \tag{4}$$

Recall that the coefficient of the second term of (4) is ω^2 and using the relations among frequency, angular frequency and period, we get

$$\omega = \sqrt{\frac{g}{R}} = 2\pi f = \frac{2\pi}{T}. \tag{5}$$

Finally, the period of the rock's motion is

$$T = 2\pi\sqrt{\frac{R}{g}} = 2 \times 3.14 \times \sqrt{\frac{6.4 \times 10^6 m}{9.8m/s^2}} = 5 \times 10^3 sec \sim 83min\, 20sec. \tag{6}$$

(b) (10 points) Now imagine a wooden cube, $10 \times 10 \times 10cm^3$ in size. The density of wood is $500kg/m^3$, so this cube floats in water. Suppose that it floats with one of its faces parallel to the water surface. You push this cube slightly down, to submerge 3/4 of its volume in the water, and then let go. The cube starts oscillating vertically around its equilibrium position. What is the frequency and amplitude of this oscillation ? (Unlike the previous question, you can do this experiment at home !)

SOLUTION: This is similar to the above problem, we are supposed to find an equation for oscillation like (4). Then, read out the physical quantities such as angular frequency or period. First, we assume the wooden cube stays parallel to water surface gently moving vertically up and down during the motion. Then, mark the middle of the height which is equal to the water level when it's untouched (you can find why from the given density of the cube, it is exactly half of water density and the cube will be submerged exactly half of its volume). Push the cube so that 3/4 of the box is submerged in the water and then, let it go making it oscillate. Notice that at this moment, the water level is $2.5cm$ above the mark in the middle ($\frac{3}{4} \times 10cm - 5cm = 2.5cm$). We can read out the amplitude of this oscillation, not surprisingly $A = 2.5cm$ because it is the maximum displacement from the equilibrium of the motion. Now, the equation of motion: there are two forces action on the box, gravity and the buoyant force. Let's say the distance from the water level to the midpoint of the height at an arbitrary moment be x (for example, x in the beginning we pushed it down to 3/4 submerged is $-2.5cm$). Then, the buoyant force at this moment becomes

$$F_b(x) = \rho_w g(5cm - x)(10cm)^2 \tag{7}$$

where $\rho_w = 1,000kg/m^3$ is the density of water. The gravity is trivially

$$F_g(x) = -\rho_c g(10cm)^3 \tag{8}$$

$\rho_c = 500kg/m^3$ is the density if wood cube. Now, we are ready to set up the net force equation,

$$\begin{aligned} F_{net} = ma &= m\ddot{x} = F_b + F_g \\ &= \rho_w g(5cm - x)(10cm)^2 - \rho_c g(10cm)^3 = -\rho_w g(10cm)^2 x \end{aligned} \tag{9}$$

where $m = \rho_c(10cm)^3$ is the mass of the wood cube. You can massage (9) to find

$$\ddot{x} + \frac{g}{5cm}x = 0. \tag{10}$$

Here, we used the fact $\rho_c = \frac{1}{2}\rho_w$ and $a = \ddot{x}$. Read out the frequency of this motion from the coefficient of the second term in (10),

$$\omega^2 = \frac{g}{0.05m} \rightarrow f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{9.8m/s^2}{0.05m}} = 2.23sec^{-1}(Hz). \tag{11}$$

Pictures are always helpful but, I have limited softwares to make documents at home. Hopefully, you guys understand the solution very well without the illustrations. Let me know if you find anything wrong in this solution. I will have my office hour on Wednesday (12/14/05)10 - 11am. Don't hesitate to ask me any questions through mail.

Good luck on your final!!