4. (30 points) Miracle above the Vatican

In the climax of the novel *Angels and Demons* by Dan Brown, the hero, Robert Langdon, escapes the anti-matter bomb explosion by jumping from a helicopter at about 10,000 feet with a makeshift parachute. He is saved by a 2x4 sq. yard (approximately 8 m\(^2\)) canopy, which he holds with his bare hands. The canopy provides enough drag force so that Robert could survive the fall into the Tiber river, recover, and bring the bad guy to justice. I am not spoiling it for you too much, am I? Here, we will analyze whether this miraculous escape is plausible (never mind the radiation from the bomb).

(a) If Robert’s mass is 80 kg, the drag coefficient for the canopy is \( C = 1 \) and the air density is 1.2 kg/m\(^3\), estimate Robert’s terminal speed.

(b) Robert was an avid swimmer and diver, which helped him survive the high-speed dip into Tiber. Ignoring air resistance, what elevation would a human need to jump from to hit the water with the same speed?

(c) What is the ratio of Robert’s terminal speeds with and without the canopy? If Robert falls face-down, we can estimate that his cross sectional area is about 1 m\(^2\). In the beginning of the book, Robert learns from a prominent physicist (director of CERN, the particle physics lab in Geneva, Switzerland) that “One square yard of drag reduces one’s speed by 20%”. How accurate is this number?

(d) If Robert reaches the terminal speed falling face-down without the parachute first (i.e. with cross sectional area of 1 m\(^2\)) and then spreads the canopy to cover 8 m\(^2\), what initial drag force would he experience? Would he be likely to hold on?

### Solution

(a) The terminal speed is reached when the drag force, directed upward, balances downward gravity force:

\[
mg = F_{\text{drag}} = \frac{1}{2}C\rho_{\text{air}}Av^2
\]

where \( A \) is the surface area. For \( A = 8 \text{ m}^2 \), the terminal speed is

\[
v = \sqrt{\frac{2mg}{C\rho_{\text{air}}A}} = 13 \text{ m/s}
\]

This is not too high.

(b) When a body falls from elevation \( h \), starting at rest, the relationship between \( h \) and the final velocity can be determined either from kinematics, or from energy conservation. We derived both expressions in class:

\[
\frac{v^2}{2} = gh
\]

So to reach the speed \( v = 13 \text{ m/s} \), a person would need to jump from the elevation

\[
h = \frac{v^2}{2g} = 8 \text{ m}
\]

(to be strict, this would be the elevation of Robert’s gut, that is, his center of mass). For a diver who routinely jumps from a 10 m platform, this drop should not be a problem. In fact, Brown describes how Tiber water was “frothy and air-filled”, softening the impact. That makes for compelling fiction, but probably unnecessary from physics reasons.

Now, holding on to the tarp and then maneuvering it into the river in a 2-mile fall, all the while dodging the gamma rays from the anti-matter annihilation... that’s impressive.

(c) The exact quote from Mr. Max Kohler was actually “One square yard of drag will slow a falling body almost twenty percent.”. It’s a bit odd that a
German physicist would use yard as a unit, so we will convert to meters instead (and for our purposes, a 20% difference between a square yard and a square meter won’t matter too much). We also assume that Kohler is talking about terminal speed – since anything else (e.g., acceleration, time) makes little sense.

Eq. (1) provides a general expression for the terminal velocity. For equal values of \( C, \rho, \) and \( m, \) the ratio of velocities is inversely proportional to the ratio of areas:

\[
\frac{v_0}{v} = \sqrt{\frac{A}{A_0}}
\]

So if \( A_0 = 1 \text{ m}^2 \) (prone person) and \( A = 8 \text{ m}^2 \) (a person holding a canopy above him), the ratio of terminal speeds is

\[
\frac{v_0}{v} = \sqrt{8} \approx 3
\]

Fortunately for Langdon, the reduction is much larger than 20% per additional square meter. Though it does not scale linearly with area, a one square yard of area in addition to the body’s own area of about one square yard would reduce the terminal speed by \( \sqrt{2} \). Not exactly 20%, but close enough for a fiction novel.

(d) For area \( A_0 = 1 \text{ m}^2 \), the terminal speed is determined by Eq. (2)

\[
v_0 = \sqrt{\frac{2mg}{C \rho_{\text{air}} A_0}} \tag{3}
\]

If Robert opens the canopy at that moment, increasing the area to \( A = 8 \text{ m}^2 \), the drag force would be

\[
F_{\text{drag}} = \frac{1}{2} C \rho_{\text{air}} A v_0^2
\]

Plugging in the expression for \( v_0 \) from Eq. (3) (here it is convenient to do this algebraically), we find

\[
F_{\text{drag}} = mg \frac{A}{A_0} = 8mg = 6.3 \text{ kN}
\]

That’s equal to 8 body weights. Such strong sudden jolt is usually spread over the entire body and dampened by the parachute harness a sky diver would normally wear. It’s unlikely a Harvard prof, even as athletic as Langdon, would be able to hold on bare-handed, but stranger things can happen in a fiction novel...

5. (40 points) The Great Pyramid of Gizeh

The Great Pyramid (also known as Pyramid of Khufu or Cheops in Greek) at Gizeh, Egypt, when first erected (it has since lost a certain amount of its outermost layer) was about 150 m high and had a square base of edge length 230 m. It is effectively a solid block of stone of density about 2.5 g/cm³.

(a) What is the minimum amount of work required to assemble the pyramid, if the stone is initially at ground level?

(b) Assume that a slave employed in the construction of the pyramid had a food intake of about 1500 Cal/day (1 Cal = 4182 joules). The Greek historian Herodotus reported that the job took 100,000 slaves 20 years. What was the minimum efficiency of a slave (defined as work done toward pyramid-building divided by energy consumed)?

(c) The blocks were probably raised to the top of the pyramid along the sides by a system of ropes, perhaps with a pulley located at the top. The average mass of the block in the pyramid is 2500 kg. Suppose that to raise a block, the slaves place it on a cart which reduces the coefficient of friction between the block and the side of the pyramid to about \( \mu = 0.1 \). Estimate the amount of work needed to raise a single block to the top of the pyramid. If each slave can pull with a force of 500 N (about 110 lbs), how many slaves are needed to raise a single block without any mechanical advantage device (e.g. a block-and-tackle) ?

5. Solution

(a) If all the stone is initially at ground level, it must be raised to its position in the pyramid. The work required to this is the total work needed to raise
the collection of small pieces to their respective elevations. This can be found by integration:

\[ W = \int dmgz = \int \rho gz dV \]

where \( dm = \rho dV \) is the mass of a small piece located at elevation \( z \). The integration is over the volume of the pyramid. The volume element is the area of the square at a height \( z \) times \( dz \), the differential of height. The square has side \( s \) at \( z = 0 \) and side \( 0 \) at \( z = h \). The width of the square decreases linearly with height, so the width and area at height \( z \) is given by

\[ w(z) = s \left( 1 - \frac{z}{h} \right) \]
\[ A(z) = s^2 \left( 1 - \frac{z}{h} \right)^2 \]

The volume element \( dV \) is given by \( dV = A(z) \, dz \). We can now perform the integral. Expanding the polynomial in \( z \)

\[ W = \rho gs^2 \int_0^h \left( z - 2\frac{z^2}{h} + \frac{z^3}{h^2} \right) \, dz \]

This is a simple integral to perform:

\[ W = \rho gs^2 h^2 \left( \frac{1}{2} - \frac{2}{3} + \frac{1}{4} \right) = \frac{1}{12} \rho gs^2 h^2 \]

Plugging in the values for these constants, we get the amount of work required to erect the pyramid

\[ W = 2.43 \times 10^{12} \text{ Joules} \]

(b) The slaves employed in building this pyramid consumed 1500 Calories per day, which is \( 6.3 \times 10^6 \) joules per day. With 100,000 slaves working for 20 years, this is 730 million slave-days of work to build the pyramid. The total energy the slaves spent is thus \( 4.6 \times 10^{15} \) joules. The efficiency thus implied is low, \( \epsilon = 5.3 \times 10^{-4} \). This does not necessarily reflect a low intrinsic efficiency, since the slaves undoubtedly expended most of their energy on activities other than lifting the stone blocks to their final position.

(c) There are 4 forces acting on a block: the force pulling it upward \( F_{\text{pull}} \), the friction force \( F_{\text{fr}} \), gravity force \( mg \), and the normal force \( N \) (see the FBD below).

The force \( F_{\text{pull}} \) is smallest when the block is moving upward with zero acceleration. The force can be found from the force balance equations:

\[ F_{\text{pull}} = F_{\text{fr}} + mg \sin \alpha \]
\[ F_{\text{fr}} = \mu N = mg \cos \alpha \]

so

\[ F_{\text{pull}} = mg(\mu \cos \alpha + \sin \alpha) = 21 \text{ kN} \tag{4} \]

That would require 42 slaves to pull without any force amplification (mechanical advantage) devices.

The work done by this force is

\[ W = F_{\text{pull}} L \]

where \( L = h / \sin \alpha \) is the distance along the side of the pyramid. Plugging that into Eq. (4), we get

\[ W = mgh \left( 1 + \frac{\mu}{\tan \alpha} \right) = mgh \left( 1 + \frac{\mu s}{2h} \right) = 4 \text{ MJ} \]