1. (20 points) **Blitz**

This is a set of simple questions to warm you up. The problem consists of ten questions, 2 points each.

1. Two hockey players skate toward each other with equal speeds of 10 m/s (relative to the ice). What is the speed of one of the skaters relative to the other?

   *Answer:* 20 m/s

2. Circle correct answer. The coefficient of static friction is usually

   (a) smaller than the coefficient of kinetic friction.
   (b) equal to the coefficient of kinetic friction.
   (c) larger than the coefficient of kinetic friction.

   *Answer:* (c)

3. Circle correct answer. When the net torque acting on a particle is zero, the angular momentum of the particle

   (a) increases
   (b) decreases
   (c) is conserved
   (d) All of the above
   (e) None of the above

   *Answer:* (e)

4. Circle correct answer. The pressure on the bottom of a 10 m deep lake is approximately

   (a) Zero
   (b) 0.1 atm
   (c) 1 atm
   (d) 2 atm
   (e) 10 atm

   *Answer:* (d) (1 atm from air and 1 atm from water)

5. Circle correct answer. Why does a real pendulum “runs down” and eventually stops?

   (a) It looses its energy due to work done by dissipative forces
   (b) It looses its energy due to work done by gravitational forces
   (c) It looses its angular momentum due to gravitational torques
   (d) It is too damp

   *Answer:* (a)

6. Circle correct answer. An astronomer, watching a distant star that is moving away from Earth, observes light that is

   (a) Same color as light really emitted by the star
   (b) Redder than the light really emitted by the star
   (c) Bluer than the light really emitted by the star

   *Answer:* (b)

7. Circle correct answer. Two sound waves of nearly equal frequencies are played simultaneously. What is the name of the acoustic phenomenon you hear if you listen to these two waves?

   (a) Beats
   (b) Interference
   (c) Resonance
   (d) Damped oscillation
   (e) None of the above

   *Answer:* (a)
(b) Diffraction
(c) Harmonics
(d) Interference

Answer: (a)

8. Circle correct answer. The ideal gas model is valid if which of the following conditions is true?

(a) The gas density is low
(b) The gas density is high
(c) The temperature is low
(d) The gas pressure is large
(e) All of the above
(f) None of the above

Answer: (a)

9. Circle correct answer. What additional kind of energy makes \( C'_{V} \) larger for a diatomic gas than for a monatomic one?

(a) Charismatic energy
(b) Translational energy
(c) Heat energy
(d) Rotational energy
(e) Solar energy

Answer: (d)

10. Explain why the pressure in a gas increases when its temperature is raised at constant volume

Answer: As the temperature increases, the kinetic energy, and therefore, the average speed of the molecules increase. For a fixed volume, it means molecules collide with the walls more often, and transfer larger momentum to the walls in each collision. Both effects increase the average force on the walls, and hence pressure.

2. (30 points) Ski Jump

Ski jumping is a nordic sport to be featured in the upcoming Winter Olympics in Turin. Imagine an athlete sliding down a 45° ski ramp of height \( H = 140 \text{ m} \), starting at rest. The ramp flattens out to horizontal direction at elevation \( h = 70 \text{ m} \), where the skier takes off (see picture). The coefficient of friction between the skis and the ramp is \( \mu = 0.1 \). You can ignore the length of the flat section at the bottom of the ramp, as well as the air resistance and any lifting force in the air.

(a) (10 points) What is the speed of the skier at the bottom of the ramp (just before takeoff) ?

(b) (10 points) What distance \( s \) will the skier travel and how long will he be in the air ?

(c) (5 points) What is the speed of the skier when he lands ?

(d) (5 points) In the Olympics, the typical jumps can be as long as 200 m. If your calculation in part (b) deviates significantly from this distance, what assumption we made was likely incorrect ?

2. Solution

(a) We can find the final velocity of the skier at the bottom of the ramp using work-energy theorem. The initial energy at the top of the ramp is purely potential

\[
E_{\text{top}} = mgH
\]

and the final energy is a combination of potential and kinetic:

\[
E_{\text{bottom}} = mgh + \frac{1}{2}mv^2
\]

Since there is a friction force on the ramp, the total mechanical energy is not conserved, but instead the change in energy is the work of the friction force:

\[
E_{\text{bottom}} - E_{\text{top}} = W_{\text{fr}} = -F_{\text{fr}}L
\]

where \( F_{\text{fr}} = \mu N \) is the magnitude of the friction force, and \( L = (H - h)/\sin \alpha \) is the length of the ramp.

To find the friction force, we need to draw the free-body diagram and compute the normal force \( N \)
The normal force is balanced in the Y direction by the projection of the gravity:

\[ N = mg \cos \alpha \rightarrow F_r = \mu mg \cos \alpha \]

Finally, we can solve for velocity at the bottom:

\[ \frac{mv^2}{2} + mgh - mgH = -\mu mgL \]

\[ v = \sqrt{2g(H-h)\left(1 - \frac{\mu}{\tan \alpha}\right)} = 35 \text{ m/s} \tag{1} \]

You can also find the velocity using acceleration. In X direction,

\[ a_x = \frac{mg \sin \alpha - F_r}{m} = g(\sin \alpha - \mu \cos \alpha) \]

And then using the standard formula

\[ v_f^2 - v_i^2 = 2a_x \Delta x \]

and \( v_f = v, v_i = 0, \Delta x = L = (H-h)/\sin \alpha \), we get

\[ v^2 = 2g(H-h)\frac{\sin \alpha - \mu \cos \alpha}{\sin \alpha} \]

which clearly simplifies to Eq. (1).

(b) Now we will use the projectile motion. The initial velocity of the skier is horizontal. The horizontal distance he covers is given by

\[ s = vt_{\text{flight}} \tag{2} \]

where \( t_{\text{flight}} \) is the time he is in the air. Since his initial vertical velocity is zero, the elevation \( h \) is related to the flight time:

\[ h = \frac{gt_{\text{flight}}^2}{2} \]

We solve this first for time:

\[ t_{\text{flight}} = \sqrt{\frac{2h}{g}} = 3.8 \text{ sec} \]

Then, plugging into Eq. (2), we find the flight distance

\[ s = vt_{\text{flight}} = 133 \text{ m} \]

(c) It is easiest to find the speed of the skier when he lands using the energy conservation. We ignore air resistance, therefore

\[ E_{\text{land}} = \frac{mv_{\text{land}}^2}{2} = E_{\text{bottom}} = \frac{mv^2}{2} + mgh \]

\[ v_{\text{land}} = \sqrt{v^2 + 2gh} = 51 \text{ m/s} \]

That’s very high, and in fact could easily be lethal. So obviously, our calculation is too simple.

(d) In fact, the Olympic skiers routinely record jumps of over 200 m, and they land at relatively modest speeds. The reason for this discrepancy is air resistance. In horizontal direction, it slows the skier down, so naively it would decrease the range. But a more important effect happens in the vertical direction, where air resistance provides a lift. In other words, air slows down skier’s fall, making the flight longer, and reducing the landing speed. In fact, if you watch this even in the Olympics, you may notice that the jumpers orient their skis and the bodies so that they act like wings. The Flying Finns are pretty good at this!

3. (35 points) **Fountains of Peterhof**

Peterhof, now known as Petrodvorets, was built on the shores of the Baltic outside of St.Petersburg by Peter the Great as his summer residence. It is famous for its luxurious palaces, designed by the best Italian architects of the 18th century, and its magnificent canals, water cascades, and fountains. The largest fountain, the Samson, features a 20 m high vertical water jet.

(a) (10 points) With what velocity does the water exit the nozzle of the fountain?

(b) (10 points) What water pressure is required for the fountain to operate?

(c) (5 points) The nozzle diameter is about 4 cm. What is the water flow rate?

(d) (10 points) What is the diameter of the jet 10 m from the nozzle? Notice that the water flows upward from the nozzle of the fountain.
3. Solution

(a) This problem is a fairly straightforward application of Bernoulli and continuity equations. First, the height of the jet is related to the velocity through the Bernoulli’s equation:

\[ p_a + \frac{\rho v^2}{2} = p_a + \rho g H \]

where \( \rho \) is the density of water and \( H = 20 \) m is the full height of the jet. The external atmospheric pressure \( p_a \) is the same at the nozzle and at the top of the jet. Therefore, the nozzle velocity is

\[ v = \sqrt{2gH} = 20 \text{ m/s} \]

That is the same vertical velocity any object (e.g. a ball) would need to go up to elevation \( h = 20 \) m.

(b) Again, Bernoulli’s equation tells us the pressure required to push the water with velocity \( v \) if it were initially at rest:

\[ p = p_a + \frac{\rho v^2}{2} \approx 3 \times 10^5 \text{ Pa} = 3 \text{ atm} \]

where \( p_a = 1 \) atm and \( \rho = 1000 \text{ kg/m}^3 \). So the gauge pressure is 2 atm. This is the pressure in a large pipe that feeds the fountain. The interesting feature of Peterhof fountains is that they are gravity-fed, i.e. the pressure is provided by the natural springs, which run off the local hills.

(c) The flow rate through the nozzle is

\[ R = Av = \frac{\pi d^2}{4} v = 0.025 \text{ m}^3/\text{s} = 7 \text{ gpm} \]

(d) This is a combination of the continuity and Bernoulli’s equations. At the elevation \( h_2 = 10 \) m, the velocity of the water can be computed from the Bernoulli’s equation:

\[ p_a + \rho gh_2 + \frac{\rho v_2^2}{2} = p_a + \frac{\rho v_2^2}{2} = p_a + \rho g H \]

\[ v_2 = \sqrt{2g(H - h_2)} = 14 \text{ m/s} \]

The continuity equation requires that the flow rates at two points of the jet are equal:

\[ R = Av = A_2v_2 \]

or

\[ \pi d_2^2 = \pi d_1^2 \frac{v_2}{v_1} \]

which means the diameter of the jet at elevation \( h_2 = 10 \) m is

\[ d_2 = d_1 \sqrt{\frac{v_1}{v_2}} = 5 \text{ cm} \]

4. (30 points) To the other side of the Earth

Imagine a straight tunnel bored through the center of the Earth from one point on the surface to the other side. We’ll assume that the tunnel is carefully built to withstand large temperatures and pressures in the middle of the Earth, and it is evacuated well enough to ignore air resistance inside the tunnel. If you drop a rock into this tunnel, it will experience a gravity force directed straight toward the center of the Earth. Calculations show that the weight of the rock inside the tunnel depends linearly on the distance \( r \) from the center of the Earth: \( W(r) = mgr/R \) where \( m \) is the mass of the rock, \( g = 10 \text{ m/s}^2 \), and \( R = 6400 \text{ km} \) is the radius of the Earth.

(a) (15 points) How long would it take for the rock to reach the other side of the tunnel?

(b) (10 points) Find the velocity and acceleration of the rock at the center of the Earth

(c) (5 points) Find the velocity of the rock when it reaches the other end of the tunnel

4. Solution

(a) This was our problem of the week. The first thing to do here is to realize that the standard kinematic equations for constant acceleration do not apply here, since the acceleration is not constant. If the weight of the rock (or gravitational force) varies linear with the distance from center, then the acceleration also varies linearly:

\[ a = \frac{g r}{R} \]

The acceleration always points to the center of the Earth. Let’s call the distance to the center \( y \). If the rock is in an evacuated tunnel above the center of the Earth, \( y > 0 \), but the acceleration points downward, so \( a_y < 0 \). If the rock is past (below) center, then \( y < 0 \),
but \( a_y > 0 \). So the acceleration is always proportional to \( y \) and has an opposite sign:

\[
a_y = -\frac{g}{R}y 
\]

(3)

Compare this to the acceleration experienced by a mass on a spring:

\[
x = -\frac{k}{m}x 
\]

For a spring, we know that this equation describes the simple harmonic motion with angular frequency \( \omega = \sqrt{k/m} \). Likewise, Eq. (3) is the equation for simple harmonic motion with angular frequency

\[
\omega = \sqrt{\frac{g}{R}}
\]

and period

\[
T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{R}{g}} = 5000 \text{ sec} = 84 \text{ min}
\]

Incidentally, the same formula also gives the period of one revolution for a low-flying Earth satellite. Now, when the rock gets to the other side of the Earth, it would travel for half of the period; therefore, the time of the journey would be

\[
t = T/2 = 42 \text{ min}
\]

(b) It is obvious from Eq. (3) that at the center of the Earth \((y = 0)\) the acceleration is zero. But what about speed? If the position of the rock is described by

\[
y(t) = R \cos \omega t
\]

then the velocity as a function of time is

\[
v(t) = \frac{dy}{dt} = -R\omega \sin \omega t
\]

(4)

The center of the Earth corresponds to \( t = T/4 = 21 \text{ min} \) (you can see that it automatically means \( y(T/4) = 0 \)), and the velocity at the center is

\[
v_{\text{center}} = v(T/4) = -R\omega = -\sqrt{gR} = -8 \text{ km/s}
\]

Incidentally, this is also the formula for the velocity of the satellite in orbit (well, actually, this is not an accident!) The \(-\) sign here means that the rock is moving past the center and away from the top of the tunnel.

(c) When the rock reaches the end of the tunnel, \( t = T/2 \), which according to Eq. (4) means \( v(T/2) = 0 \). This is obvious. During the motion, the rock accelerates towards the center, then flies past it, and the gravity starts to slow it down again. The other side of the Earth corresponds to the other extreme of the oscillations. If there is no friction in the tunnel, the mechanical energy is conserved, so if the rock starts with zero velocity on one end of the tunnel, it would reach the other end with zero velocity. There, it would pause momentarily, and then fall back into the tunnel to continue the oscillations.

5. (25 points) Piano tuning

A piano is tuned so that the middle C key emits sound at a frequency \( f_{C4} = 262 \text{ Hz} \) and C-above-middle C at \( f_{C5} = 524 \text{ Hz} \). The piano tune uses an equal temperament spacing between notes. The white keys of its middle octave consist of middle C (key C4); D4 (1 step above middle C); E4 (2 steps); F4 (2.5 steps); G4 (3.5 steps); A4 (4.5 steps); B4 (5.5 steps); and C5 (6 steps). Each step causes the frequency to be multiplied by a fixed factor.

(a) (10 points) Find the frequencies of A4 and E4

(b) (5 points) Find the beat frequency between the second harmonic of G4 (fundamental frequency of 392.6 Hz) and the third harmonic of middle C4. (Pianos are tuned by listening for such beats.)

Stringed instruments are tuned (for compositions in the key of C) so that this beat frequency is zero, producing a smoother tone. When a composition in a different key is played, the stringed instrument can be retuned for that key, which would be impossible for the piano.

(c) (10 points) In a 1931 Steinway grand piano, key A4 uses steel strings of length \( l = 40.5 \text{ cm} \) and diameter \( d = 0.99 \text{ mm} \). What is the typical tension of these strings? The density of steel is \( \rho = 7.8 \text{ g/cm}^3 \).

5. Solution

(a) Equal temperament spacing described above means that for each step the frequency is multiplied by
a fixed amount. For instance,
\[ f_{D4} = \delta f_{C4} \]
\[ f_{E4} = \delta f_{D4} = \delta^2 f_{C4} \]
etc. Here \( \delta \) is a multiplier that corresponds to 1 step. By induction (i.e. continuing this logic to note C5),
\[ f_{C5} = \delta^6 f_{C4} \]
which lets us compute the multiplier:
\[ \delta = \left( \frac{f_{C5}}{f_{C4}} \right)^{1/6} = 2^{1/6} \approx 1.122 \]
So each step corresponds to a 12% increase in frequency. Thus, we can compute each frequency in the octave as
\[ f = \delta^n f_{C4} \]
(5)
or
\[ \log f = n \log \delta + \log f_{C4} \]
where \( n \) is the number of steps separating a given note from C4. This last equation shows that the notes are linearly spaced on the log scale. So the musical scale is logarithmic.

Using Eq. (5), we can now compute the frequencies of A4 \((n = 4.5)\) and E4 \((n = 2)\):
\[ f_{A4} = \delta^{4.5} f_{C4} = 440.6 \text{ Hz} \]
\[ f_{E4} = \delta^2 f_{C4} = 330.1 \text{ Hz} \]
(b) The 2nd harmonic of G4 has the frequency
\[ f_{2,G4} = 2f_{G4} = 785.2 \text{ Hz} \]
The 3rd harmonic of C4 has the frequency of
\[ f_{3,C4} = 3f_{C4} = 786 \text{ Hz} \]
and the beat frequency is
\[ f_{\text{beat}} = 3f_{C4} - 2f_{G4} = 0.8 \text{ Hz} \]
It is fairly easy to recognize these beats at the rate of about once per second.

(c) The tension of the string is related to the velocity of sound as
\[ v = \sqrt{\frac{T}{\mu}} \]
where \( \mu = \rho A \) is the linear mass density and \( A = \pi d^2 / 4 \) is the cross sectional area of the string. The piano strings are fixed at both sides, so the first harmonic of a string satisfies the closed-closed configuration for the standing wave:
\[ \lambda_1 = 2l \]
where
\[ \lambda_1 = \frac{v}{f_1} \]
is the wavelength of the 1st harmonic, and \( f_1 \) is its frequency. From part (a) we know \( f_1 = 440.6 \text{ Hz} \) (for the piano tuned to C4 key). Putting all of this together, we can solve for tension:
\[ T = 4\mu l^2 f_{A4}^2 = \pi \rho (dl f_{A4})^2 = 765 \text{ N} \]

6. (20 points) Quick Thanksgiving Dinner
It is 4 pm on a Thanksgiving evening, and you need to prepare a holiday meal pronto. A quick trip to Safeway before they close earns you a 22 lb \((m = 10 \text{ kg})\) bird. To cook it as quick as possible, you stick it, still frozen, into the microwave. Your fancy microwave can be programmed to run a defrost cycle first and then cook the meat on “High” power setting. How would you program it ? For the purpose of this calculation, we will assume that the meat consists mostly of water. The bird was initially at \( T_i = -10^\circ \text{C} \), and “Joy of Cooking” recommends cooking poultry until the meat reaches a temperature of \( T_f = 82^\circ \text{C} \) \((180^\circ \text{F})\). The specific heat of water is \( c_{\text{water}} = 4200 \text{ J/kg/K} \), specific heat of ice is \( c_{\text{ice}} = 2100 \text{ J/kg/K} \), and the latent heat of fusion for water is \( L_f = 333,000 \text{ J/kg} \).

(a) (10 points) The “Defrost” setting delivers the average thermal power of \( P_d = 300 \text{ W} \) to the turkey (it actually turns the microwaves on and off, to allow the heat to spread uniformly). How long do you need to run the microwave on “Defrost” before all the ice melts ?
(b) (10 points) After “Defrost” cycle is complete, the microwave turns to “High”, delivering thermal power of $P_h = 1$ kW. The cooking stops when the internal temperature reaches $T_f = 82^\circ$C. How long does that take?

Enjoy your delicious meal!

6. Solution
(a) To melt the ice, we need to

1. Raise the temperature of ice from $T_i = -10^\circ$C to $T_m = 0^\circ$C. This requires

   $$ Q_w = mc_{\text{ice}} (T_m - T_i) = 2.1 \times 10^5 \text{ J} $$

   of heat.

2. Melt the ice. This requires

   $$ Q_m = mL_f = 3.33 \times 10^6 \text{ J} $$

   of heat.

The total heat required is

$$ Q_{\text{defrost}} = Q_w + Q_m = 3.54 \times 10^6 \text{ J} $$

The time required to deliver this much heat on defrost setting is

$$ t_{\text{defrost}} = \frac{Q_{\text{defrost}}}{P_d} = 3.3 \text{ hours} $$

Clearly, this does not happen very fast.

(b) To cook the bird, we need to raise its temperature from $T_m = 0^\circ$C to $T_f = 82^\circ$C. This requires

$$ Q_{\text{cook}} = mc_{\text{water}} (T_f - T_m) = 3.44 \times 10^6 \text{ J} $$

of heat, and takes

$$ t_{\text{cook}} = \frac{Q_{\text{cook}}}{P_h} \approx 1 \text{ hour} $$

on “high” setting. So in total, you can have a turkey cooked in a microwave in a bit under 4.5 hours. Most of the time though is spent on defrosting instead of cooking, so if you really need to cook the meat fast, do not buy it frozen!

7. (40 points) Diving dangerously

(a) (10 points) If a scuba diver fills his lungs to full capacity of 5.5 L when 10 m below the surface, to what volume would his lungs expand if he rose to the surface without breathing out? Is this advisable? Assume that the temperature of the air inside lungs remains constant at $T \approx 37^\circ$C.

(b) (15 points) How much work would be done by the air in the lungs during the ascend?

(c) (15 points) If during the ascend the diver releases some of the air, to keep the volume of his lungs constant at 5.5 L, what mass of the air is released? Use $M = 29 \text{ g/mol}$ for the molar mass of air.

7. Solution

(a) Pressure increases linearly with depth below the surface

$$ P_0 = P_{\text{atm}} + \rho gd, \quad (6) $$

where, $P_{\text{atm}} = 1.01 \times 10^5 \text{ Pa}$ is the atmospheric pressure, $\rho = 10^3 \text{kg/m}^3$ is the density of water, and $g = 9.8 \text{m/s}^2$ is gravitational acceleration. Since the diver is warm-blooded, her core temperature will not change much during the rise to the surface, so the pressure $P$ and volume $V$ of the gas in her lungs obeys $PV = \text{const}$. If the initial volume is $V_0 = 5.5\text{L}$, this implies a final volume of

$$ V_f = V_0 \frac{P_{\text{atm}} + \rho gd}{P_{\text{atm}}} = 10.8 \text{L}. $$

The volume of the lungs would double. Since this maneuver would put substantial stress on the ribcage, risking fracture, it would not be advisable.

(b) The work done by the expanding gas is

$$ W = \int_{V_0}^{V_f} P(V) dV $$

At constant temperature,

$$ PV = P_0 V_0 = \text{const} $$

and

$$ W = P_0 V_0 \int_{V_0}^{V_f} \frac{dV}{V} = P_0 V_0 \ln \left( \frac{V_f}{V_0} \right) = 540 \text{ J} $$
Here we will write ideal gas law in a less-commonly used form:

\[ PV = \frac{m}{M}RT \]

where \( M = 29 \times 10^{-3} \) kg/mole is the molar mass of air. Normally, we keep the mass of the gas (and the number of moles) constant, and look at changes of pressure, volume, or temperature. In this case, we keep the volume and temperature constant, but change mass \( m \) to match the decrease in pressure.

Initially, the pressure of the gas is

\[ P_0 = P_{\text{atm}} + \rho gd = 1.99 \times 10^5 \text{ Pa} \]

the volume is \( V_0 = 5.5 \text{ L} \), and temperature is \( T = 310 \text{ K} \). Therefore, the lungs contain

\[ m_0 = \frac{P_0V_0M}{RT} = 12.3 \text{ g} \]

of air. At the surface, the pressure is \( P = P_{\text{atm}} \), and the lungs should contain

\[ m_{\text{surface}} = \frac{P_{\text{atm}}V_0M}{RT} = 6.3 \text{ g} \]

of air. Therefore, the diver would release

\[ \Delta m = m_0 - m_{\text{surface}} = 6 \text{ g} \]

of air. That’s a relatively small amount (by mass), but it corresponds to significant volume.

[1] The problem posted on the web had a typo in the value of the frequency of G4 note